Negative Energy in Superposition and Entangled States

L.H. Ford*

Institute of Cosmology

Department of Physics and Astronomy

Tufts University, Medford, MA 02155

Thomas A. Roman†

Department of Mathematical Sciences

Central Connecticut State University

New Britain, CT 06050

Abstract

We examine the maximum negative energy density which can be attained in various quantum states of a massless scalar field. We consider states in which either one or two modes are excited, and show that the energy density can be given in terms of a small number of parameters. We calculate these parameters for several examples of superposition states for one mode, and entangled states for two modes, and find the maximum magnitude of the negative energy density in these states. We consider several states which have been, or potentially will be, generated in quantum optics experiments.

PACS numbers: 04.62.+v,03.65.Ud,42.50.Pq

*Email: ford@cosmos.phy.tufts.edu

†Email: roman@ccsu.edu

I. INTRODUCTION

It has been proven, beginning in the early 1960's [1], that there always exist states with negative energy density in quantum field theory. Some specific examples include the Casimir effect [2] and squeezed states [3], both of which have been experimentally realized. (Although the energy density itself is far too small to be directly measured.) Negative energy is also required for black hole evaporation, and hence for the consistency of the laws of black hole physics with those of thermodynamics. On the other hand, unrestricted amounts of negative energy could produce bizarre effects, for example, violations of the second law of thermodynamics [4, 5]. However, the same laws of quantum field theory which allow the existence of negative energy also appear to severely restrict its magnitude and duration in such a way as to prevent gross large-scale effects. These bounds are known as quantum inequalities, and quite a large body of work now exists on the subject. For some recent reviews of quantum inequalities, see Refs. [6, 7, 8]. Quantum inequality bounds have been proven, for example, for the minimally coupled scalar, electromagnetic, and Dirac fields. It should be pointed out that the potential macroscopic problems arise not because of the existence of negative energy per se, but from the arbitrary separation of negative and positive energy. It is this behavior which the quantum inequalities prohibit. Many possible configurations of separated negative and positive energy can easily be ruled out, and known permitted examples involve the subtle intertwining of the two [9]. Whether the currently known examples are representative of the general case is unknown. Hence, the study of further examples could prove useful.

Since the negative energy densities in these states are too small to be directly measurable, experiments in quantum optics may offer the best possibilities for indirect detection of negative energy. (However, see also Refs. [10, 11].) A first link between quantum optics and the work on quantum inequalities has been forged in a recent paper by Marecki [12]. For squeezed states, he proved quantum inequality-type bounds on the magnitude and duration of the squeezing.

Quantum optics has seen enormous experimental and theoretical advances in the last twenty years. This marriage of optics with quantum field theory has resulted in experiments which were formerly purely "gedanken" becoming those which are now routinely performed in the laboratory. Highly non-classical states, such as Schrödinger "cat states" and squeezed

states, have been produced and play a part in everything from quantum computers to noise reduction in laser interferometer gravitational wave detectors. The "cat states" of the electromagnetic field are superpositions of coherent states and have been created experimentally [13, 14]. The experiments which have been done so far have produced mesoscopic superpositions, in which the mean photon number is of order 10. This is somewhat short of a true Schrödinger cat state, which would be a superposition of two or more classical configurations, that is, coherent states with very large occupation numbers. More recently, there have been proposals for methods of creating superpositions of squeezed vacuum states [15].

An interesting question arises: can one start with two quantum states which do not involve negative energy and by superposing them obtain negative energy? The answer is yes; the classic standard example being the vacuum + two-particle state (for a nice discussion see Ref. [16]). Is this true for the superposition of other states as well? More generally, what effects does superposition have on negative energy? Could one also go the other way, i.e., start with two states involving negative energy and by superposing them diminish or eradicate the negative energy? In this paper, we will address such questions for several classes of states. In Sect. II, we develop some formalism for parameterizing the maximum magnitude of negative energy that can occur for states of a minimally coupled scalar field in Minkowski spacetime with either one or two modes excited. We give several examples of superpositions for a single mode in Sect. III, including superpositions of two coherent states, two squeezed vacuum states, and a coherent state with a squeezed vacuum state. In Sect. IV, we move to the two-mode case. This allows us to consider examples of entangled states involving either squeezed vacua or coherent states for the two modes. A summary of our conclusions is presented in Section V.

II. ENERGY DENSITY WITH ONE OR TWO MODES ARE EXCITED

In this paper, we will consider a massless scalar field in flat spacetime, for which the stress tensor operator is

$$T_{\mu\nu} = \varphi_{,\mu}\,\varphi_{,\nu} - \frac{1}{2} g_{\mu\nu}\,\varphi_{,\sigma}\varphi^{,\sigma}\,. \tag{1}$$

The normal-ordered energy density operator is

$$:T_{00}:= \frac{1}{2}[:\dot{\varphi}^2: + :(\nabla\varphi)^2:]\,, \tag{2}$$

where

$$\varphi = \sum_k (a_k f_k + a_k{}^\dagger f_k{}^*)\,, \tag{3}$$

with the $f_k(\mathbf{x}, t)$ being the mode functions.

A. Two Modes Excited

We wish to consider the case where all modes except for two are in the vacuum state. For the first mode, let f_1, a, and $a^\dagger$ be the mode function, annihilation operator, and creation operator, respectively, and let f_2, b, and $b^\dagger$ be the corresponding quantities for the second mode. The expectation value of the energy density in an arbitrary quantum state can be expressed as

$$\begin{aligned}\rho \;=\; \langle : T_{00} : \rangle = \mathrm{Re}\Big\{ &\langle a^\dagger a\rangle \,(|\dot f_1|^2 + |\nabla f_1|^2) + \langle a^2\rangle\,[\dot f_1^2 + (\nabla f_1)^2] + \langle b^\dagger b\rangle\,(|\dot f_2|^2 + |\nabla f_2|^2) \\ +\;& \langle b^2\rangle\,[\dot f_2^2 + (\nabla f_2)^2] + 2\langle a^\dagger b\rangle\,(\dot f_1^* \dot f_2 + \nabla f_1{}^* \cdot \nabla f_2) + 2\langle ab\rangle\,(\dot f_1 \dot f_2 + \nabla f_1 \cdot \nabla f_2)\Big\}\,. \end{aligned} \tag{4}$$

Let

$$n_1 = \langle a^\dagger a\rangle, \quad n_2 = \langle b^\dagger b\rangle, \quad R_1\, \mathrm{e}^{i\gamma_1} = \langle a^2\rangle, \quad R_2\, \mathrm{e}^{i\gamma_2} = \langle b^2\rangle, \quad R_3\, \mathrm{e}^{i\gamma_3} = \langle a^\dagger b\rangle, \quad R_4\, \mathrm{e}^{i\gamma_4} = \langle ab\rangle\,. \tag{5}$$

All of the information needed to give the two-mode energy density, Eq. (4), at a given quantum state is encoded in the above set of six amplitudes and four phases.

In the case of a traveling waves, we may take the mode functions to be

$$f_j = \frac{i}{\sqrt{2\omega_j V}} \mathrm{e}^{i(\mathbf{k}_j \cdot \mathbf{x} - \omega_j t)}\,, \tag{6}$$

where $\omega_j = |\mathbf{k}_j|$, for $j = 1, 2$ and V is a normalization volume. In this case, the mean energy density may be expressed as

$$\begin{aligned}\rho \;=\; \frac{1}{V}\Big\{& n_1\,\omega_1 + n_2\,\omega_2 + R_1\,\omega_1\,\cos[2(\mathbf{k}_1 \cdot \mathbf{x} - \omega_1 t) + \gamma_1] + R_2\,\omega_2\,\cos[2(\mathbf{k}_2 \cdot \mathbf{x} - \omega_2 t) + \gamma_2] \\ +\;& R_3\,\sqrt{\omega_1 \omega_2}\,(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)\,\cos[(\mathbf{k}_2 - \mathbf{k}_1)\cdot \mathbf{x} - (\omega_2 - \omega_1)t + \gamma_3] \\ +\;& R_4\,\sqrt{\omega_1 \omega_2}\,(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)\,\cos[(\mathbf{k}_2 + \mathbf{k}_1)\cdot \mathbf{x} - (\omega_2 + \omega_1)t + \gamma_4]\Big\}\,. \end{aligned} \tag{7}$$

We will also consider the case of a standing wave which depends upon only one space coordinate, in which case the mode functions can be taken to be

$$f_j = \frac{1}{\sqrt{\omega_j V}}\,\sin(\omega_j x)\,\mathrm{e}^{-i\omega_j t}\,. \tag{8}$$

The energy density now becomes

$$\rho = \frac{1}{V}\Big\{n_1\,\omega_1 + n_2\,\omega_2 + R_1\,\omega_1\,\cos(2\omega_1 x)\,\cos(2\omega_1 t - \gamma_1) + R_2\,\omega_2\,\cos(2\omega_2 x)\,\cos(2\omega_2 t - \gamma_1)$$
$$+\;2R_3\,\sqrt{\omega_1\omega_2}\,\cos[(\omega_2-\omega_1)x]\,\cos[(\omega_2-\omega_1)t-\gamma_3]$$
$$+\;2R_4\,\sqrt{\omega_1\omega_2}\,\cos[(\omega_1+\omega_2)x]\,\cos[(\omega_1+\omega_2)t-\gamma_4]\Big\}\,. \tag{9}$$

B. One Mode Excited

A useful special case is when only one mode is excited. In this case, we may set $n_1 = n$, $R_1 = R$, $\gamma_1 = \gamma$, and $R_2 = R_3 = R_4 = \gamma_2 = \gamma_3 = \gamma_4 = 0$. In this case, we need only the three real numbers n, R, and γ to determine the energy density in a given state. For the case of a traveling wave, we have

$$\rho = \frac{\omega}{V}\,\{n + R\,\cos([2(\mathbf{k}\cdot\mathbf{x} - \omega t) + \gamma]\} \tag{10}$$

We can see from Eq. (10) that the minimum value of ρ is

$$\rho_{min} = -\frac{\omega}{V}\,(R-n)\,, \tag{11}$$

and hence we can have negative energy density only if $R > n$. In the case of a standing wave, Eq. (9) becomes

$$\rho = \frac{\omega}{V}\,[n + R\,\cos(2\omega x)\,\cos(2\omega t - \gamma)]\,. \tag{12}$$

Again, the minimum value of ρ is given by Eq. (11).

III. SUPERPOSITIONS FOR ONE MODE

In this section, we examine some explicit examples of superpositions involving a single mode. In each case, we need only calculate the quantity $R - n$ to determine the maximum magnitude of the negative energy.

A. Superposition of Two Coherent States

First we consider a superposition of coherent states. Coherent states are eigenstates of the annihilation operator, that is

$$a|\alpha\rangle = \alpha|\alpha\rangle\,. \tag{13}$$

Let

$$\psi\rangle = N[|\alpha\rangle + \eta|\beta\rangle]\,, \tag{14}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are two different coherent states for the same mode, η is a complex number, and N is a normalization factor (see, for example, Sec. 7.6 of Ref. [17]). We also assume that the states are normalized so that $\langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1$. As a result we have that

$$\langle\psi|\psi\rangle = 1 = N^2[1 + |\eta|^2 + \eta\langle\alpha|\beta\rangle + \eta^*\langle\beta|\alpha\rangle]\,. \tag{15}$$

The coherent states are not orthonormal; their overlap integral is given by (see for example, Eq.(3.6.24) of Ref. [18]):

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)}\,. \tag{16}$$

Therefore the square of the normalization factor is

$$N^2 = [1 + |\eta|^2 + 2\mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\,\mathrm{Re}(\eta\mathrm{e}^{\alpha^*\beta})]^{-1}\,. \tag{17}$$

The mean number of particles is found to be

$$n = N^2\left[|\alpha|^2 + |\eta\beta|^2 + 2\mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\,\mathrm{Re}\big(\eta\alpha^*\beta\mathrm{e}^{\alpha^*\beta}\big)\right]\,, \tag{18}$$

and

$$\langle a^2\rangle = N^2\left[\alpha^2 + |\eta|^2\beta^2 + \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\,(\eta\beta^2\mathrm{e}^{\alpha^*\beta} + \eta^*\alpha^2\mathrm{e}^{\alpha\beta^*})\right]\,. \tag{19}$$

Let

$$\alpha = |\alpha|\mathrm{e}^{i\delta_1}\,, \quad \beta = |\beta|\mathrm{e}^{i\delta_2}\,, \quad \text{and} \quad \eta = |\eta|\mathrm{e}^{i\delta}\,. \tag{20}$$

Then the quantities n, R, and γ are functions of six real parameters, the magnitudes and phases of α, β, and η. However, one finds that only γ depends upon all six. The magnitudes n and R depend only upon the difference $\delta_2 - \delta_1$, and are hence functions of five parameters. We are primarily interested in the quantity $R - n$, which measures the maximum magnitude of the negative energy density. Hence set $\delta_1 = 0$ and write

$$F(|\alpha|, |\beta|, |\eta|, \delta_2, \delta) = R - n\,, \tag{21}$$

and let G be a five-dimensional vector given by

$$G = \left(\frac{\partial F}{\partial|\alpha|}, \frac{\partial F}{\partial|\beta|}, \frac{\partial F}{\partial|\eta|}, \frac{\partial F}{\partial\delta_2}, \frac{\partial F}{\partial\delta}\right)\,. \tag{22}$$

One may use Eqs. (21) and (22) as the basis of a numerical algorithm to search for points of maximum F and hence maximally negative energy density. Start at a random point in the five-dimensional parameter space, and compute F and G. If $F > 0$, then this choice of parameters is a quantum state with negative energy density. The components of G indicate the direction in which F is increasing most rapidly. One then moves along this direction until a local maximum of F is located. A preliminary, non-exhaustive, search located two such local maxima, at $(|\alpha|, |\beta|, |\eta|, \delta_2, \delta) \approx (0.8, 0.8, 1, 3.14, 0)$ and at $(|\alpha|, |\beta|, |\eta|, \delta_2, \delta) \approx (0, 1.61, 1, 0, 0)$. (One can trivially generate a third maximum by interchange of α and β in the latter case.) The first example corresponds to $\alpha = -\beta = 0.8$ and the second to a superposition of a coherent state and the vacuum. Interestingly, the maximum magnitude of the negative energy density is about the same in both examples, with $F = R - n \approx 0.278$, and hence $\rho_{min} \approx -0.278\,\omega/V$. We do not have an explanation as to why these two choices give the same value of $R - n$. The mean particle number is $n \approx 0.36$ in the first example and $n \approx 1.0$ in the second. This example illustrates that a superposition of two coherent states can produce negative energy density, and the maximum negative energy density arises for mean particle number of order one.

B. Superposed Squeezed Vacuum States

1. A Single-Mode Squeezed Vacuum State

We begin with a review of the features of the expectation value of the energy density in a single squeezed vacuum state. Our state is given by:

$$|\psi\rangle = |\xi\rangle \quad , \text{with } \xi = r\, e^{i\delta} , \tag{23}$$

where r is the squeeze parameter and δ is a phase parameter. The squeeze operator $S(\xi)$ is given by

$$S(\xi) = e^{\frac{1}{2}[\xi a^2 - \xi^* (a^\dagger)^2]} . \tag{24}$$

This operator is unitary since

$$S^\dagger(\xi) = S(-\xi) = S^{-1}(\xi) . \tag{25}$$

The single-mode squeezed state $|\xi\rangle$ is produced by the squeeze operator acting on the vacuum state

$$|\xi\rangle = S(\xi)|0\rangle\,. \tag{26}$$

The state $|\xi\rangle$ can be written, after some work (see Eq. (3.7.5) of Ref. [18]), in terms of the even Fock states as

$$|\xi\rangle = \sqrt{\mathrm{sech} r}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{n!}\Big[-\frac{1}{2}e^{i\delta}\mathrm{tanh} r\Big]^{n}|2n\rangle\,. \tag{27}$$

We also have that

$$\begin{aligned} S^{\dagger}(\xi)aS(\xi) &= a\,\mathrm{cosh} r - a^{\dagger}e^{i\delta}\mathrm{sinh} r\,,\\ S^{\dagger}(\xi)a^{\dagger}S(\xi) &= a^{\dagger}\mathrm{cosh} r - a\,e^{-i\delta}\mathrm{sinh} r\,. \end{aligned} \tag{28}$$

In the state $|\xi\rangle$ we have the expectation values

$$\begin{aligned} n &= \langle a^{\dagger}a\rangle = \langle 0|S^{\dagger}(\xi)a^{\dagger}aS(\xi)|0\rangle = \langle 0|S^{\dagger}(\xi)a^{\dagger}S(\xi)S^{\dagger}(\xi)aS(\xi)|0\rangle = \mathrm{sinh}^2 r\,,\\ \langle a^2\rangle &= \langle 0|S^{\dagger}(\xi)a^2S(\xi)|0\rangle = \langle 0|S^{\dagger}(\xi)aS(\xi)S^{\dagger}(\xi)aS(\xi)|0\rangle = -e^{i\delta}\mathrm{sinh} r\,\mathrm{cosh} r\,, \end{aligned} \tag{29}$$

where we have made use of Eqs. (28). Thus $R = \sinh r\ \cosh r$ and

$$R - n = \sinh r(\cosh r - \sinh r) \tag{30}$$

attains its maximum value of 0.5 as $r \to \infty$.

2. Superposition of Squeezed Vacuum States

We now calculate the energy density in a superposition of two single-mode squeezed vacuum states of the form

$$|\psi\rangle = N[|\xi\rangle + \eta| - \xi\rangle]\,, \tag{31}$$

where, for simplicity, we will choose

$$\xi = r, \qquad \delta = 0\,, \tag{32}$$

and set

$$\eta = |\eta|\,\mathrm{e}^{i\theta}\,. \tag{33}$$

In this state we have

$$
\begin{aligned}
n = \langle a^\dagger a\rangle &= N^2[\langle\xi|a^\dagger a|\xi\rangle + |\eta|^2\,\langle-\xi|a^\dagger a|-\xi\rangle + \eta\,\langle\xi|a^\dagger a|-\xi\rangle + \eta^*\,\langle-\xi|a^\dagger a|\xi\rangle] \\
&= N^2\left[\sinh^2 r(1+|\eta|^2) - 2|\eta|\,\cos\theta\,\frac{\mathrm{sech} r\,\tanh^2 r}{(1+\tanh^2 r)^{3/2}}\right],
\end{aligned} \tag{34}
$$

where we have made use of Eq. (A4) in the Appendix. A similar calculation, using Eq. (A5), yields

$$
\langle a^2\rangle = N^2\left[(|\eta|^2-1)\sinh r\cosh r + 2\,i|\eta|\,\sin\theta\,\frac{\mathrm{sech} r\,\tanh r}{(1+\tanh^2 r)^{3/2}}\right], \tag{35}
$$

and

$$
R = |\langle a^2\rangle| = N^2\left[(|\eta|^2-1)^2\sinh^2 r\cosh^2 r + 4|\eta|^2\,\sin^2\theta\,\frac{\mathrm{sech}^2 r\,\tanh^2 r}{(1+\tanh^2 r)^3}\right]^{\frac{1}{2}}. \tag{36}
$$

The normalization of our state is given by

$$
\langle\psi|\psi\rangle = 1 = N^2\,[1+|\eta|^2 + \eta\langle\xi|-\xi\rangle + \eta^*\langle-\xi|\xi\rangle]\,. \tag{37}
$$

The square of the normalization factor is then

$$
N^2 = \left[1+|\eta|^2 + 2|\eta|\,\cos\theta\,\sqrt{\mathrm{sech}(2r)}\right]^{-1}, \tag{38}
$$

where we have used Eq. (A8) of the Appendix.

From Eqs. (34) and (36), we can compute the quantity $R-n$, which gives the maximum magnitude of the negative energy density, as function of θ and r. For fixed θ, one typically finds that $R-n$ attains a maximum value for some value of r, usually of order one. A typical case of $\theta = 0$ is illustrated in Fig. 1. The case $\eta = 0$ is just the single squeezed vacuum state discussed in Sect. III B 1 . This case gives the maximum negative energy density, $R-n = 0.5$, for large r. All other values of η, corresponding to superposed squeezed vacua, give somewhat smaller amounts of negative energy density, and attain their maximum negative energy density at finite values of r.

C. Superposition of Coherent and Squeezed Vacuum States

In this subsection, we consider states of the form

$$
|\psi\rangle = N[|\xi\rangle + \eta|\alpha\rangle]\,, \tag{39}
$$

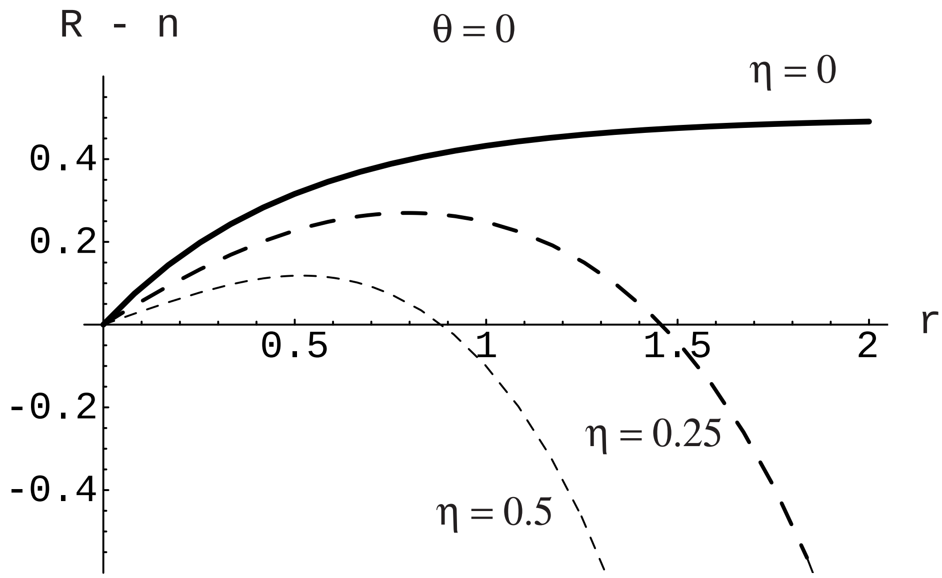

FIG. 1: The quantity $R-n$ for two superposed squeezed vacua, Eq. (31), is plotted for the case $\theta = 0$ for various values of η. The case $\eta = 0$ is the single squeezed vacuum, and gives more negative energy density than do any of the superpositions. For non-zero η, there is a maximum value for $R-n$ at a finite value of r.

where $|\xi\rangle$ is a squeezed vacuum state, and $|\alpha\rangle$ is a coherent state. We may use Eq. (A10) to find

$$N^2 = \left\{1 + |\eta|^2 + 2\sqrt{\operatorname{sech} r}\, \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\, \mathrm{Re}\left[\eta \exp\left(-\frac{1}{2}\, \mathrm{e}^{-i\delta}\, \alpha^2\, \tanh r\right)\right]\right\}^{-1} . \tag{40}$$

Similarly, we find

$$\begin{aligned} n \;=\; \langle a^\dagger a\rangle = N^2 \Big\{&\sinh^2 r + |\eta\alpha|^2 \\ -\; &2\mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sqrt{\operatorname{sech} r}\, \tanh r\, \mathrm{Re}\left[\eta\, \mathrm{e}^{-i\delta}\, \alpha^2\, \exp\left(-\frac{1}{2}\, \mathrm{e}^{-i\delta}\, \alpha^2\, \tanh r\right)\right]\Big\} \end{aligned} \tag{41}$$

and

$$\begin{aligned} \langle a^2\rangle \;=\; N^2 \Big\{&-\sinh r \cosh r\, \mathrm{e}^{i\delta} + |\eta|^2\alpha^2 + \eta\, \alpha^2\, \sqrt{\operatorname{sech} r}\, \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\, \exp\left(-\frac{1}{2}\, \mathrm{e}^{-i\delta}\, \alpha^2\, \tanh r\right) \\ +\; &\eta^*\, \sqrt{\operatorname{sech} r}\, \mathrm{e}^{-\frac{1}{2}|\alpha|^2}[(\alpha^*)^2\, \mathrm{e}^{i\delta}\, \tanh r - 1]\, \mathrm{e}^{i\delta}\, \tanh r\, \exp[-\frac{1}{2}\, \mathrm{e}^{i\delta}\, (\alpha^*)^2\, \tanh r]\Big\}, \end{aligned} \tag{42}$$

using Eqs. (A11) and (A13).

Let us consider the case where $\delta = 0$, $\eta = 1$, and α is real. In this case, $R-n$ is plotted in Fig. 2 for various values of α. Here the maximum negative energy density, $R-n \approx 0.23$

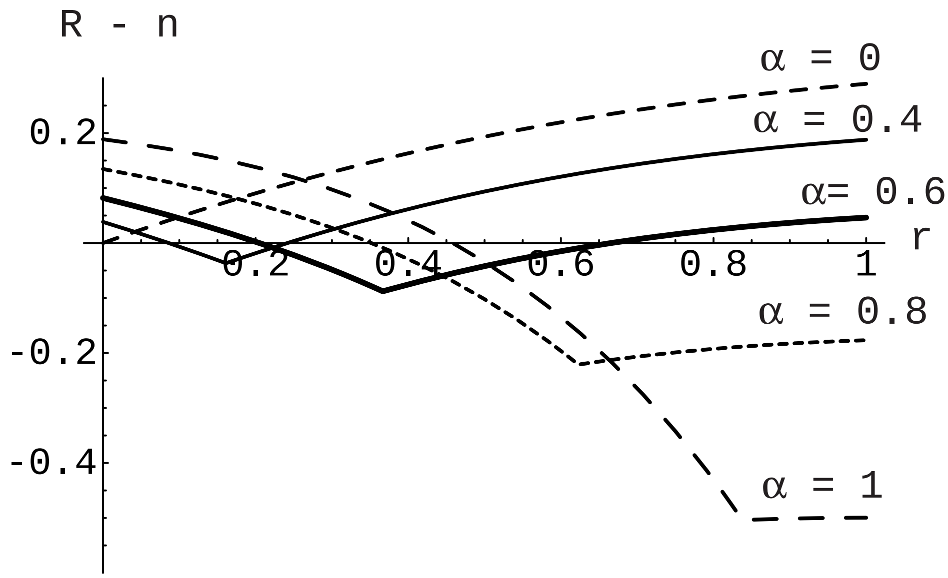

FIG. 2: Here $R-n$ is plotted for the superposition of a coherent and a squeezed vacuum state, Eq. (39) with $\eta = 1$, as a function of r for various values of α. Note that for $\alpha \neq 0$, $R-n$ can be positive, corresponding to negative energy, for both smaller and larger values of r, but has an intermediate region where there is no negative energy.

is attained for large r. Note that this state has less negative energy than does the squeezed vacuum by itself. [See Eq. (30).] For α non-zero, we find that $R-n$ initially decreases, reaches a minimum value, and then increases again. For the case $\alpha = 0.6$, for example, there is negative energy for $r < 0.2$ and again for $r > 0.65$, but not for intermediate values of r. Note that $r = 0$ is a superposition of the vacuum and a coherent state, a special case of the state treated in Sect. III A.

A limit of special interest is when $\alpha = 0$ and we have the superposition of the vacuum with a squeezed vacuum state. In this case,

$$N^2 = \left[1 + |\eta|^2 + 2\mathrm{Re}(\eta)\,\sqrt{\mathrm{sech}\, r}\,\right]^{-1}\,, \tag{43}$$

$$n = \langle a^\dagger a\rangle = N^2\,\sinh^2 r\,, \tag{44}$$

and

$$\langle a^2\rangle = -N^2\,\sinh r\cosh r\,\mathrm{e}^{i\delta}\,\left[1 + \eta^*\,(\mathrm{sech}\, r)^{\frac{5}{2}}\right]\,. \tag{45}$$

The $\alpha = 0$ curve in Fig. 2 is this limit for $\eta = 1$. If η is real and negative, $\eta = -|\eta|$, we then

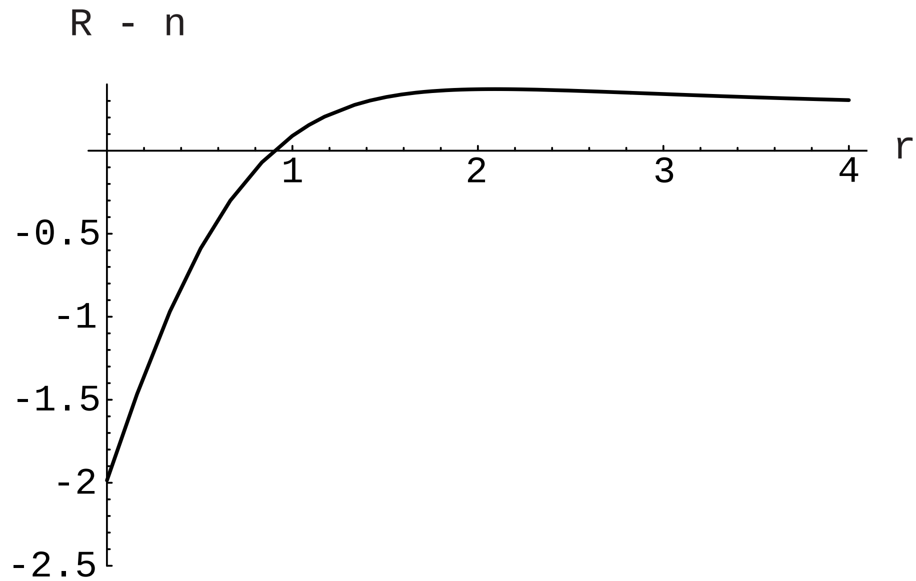

FIG. 3: Here $R-n$ is plotted as a function of r for $\alpha = 0$ and $\eta = -1$, a superposition of the vacuum and a squeezed vacuum. The maximum negative energy occurs when $r \approx 2$, where $R - n \approx 0.3$.

have

$$R = |\langle a^2 \rangle| = N^2 \sinh r \cosh r \, \left| 1 - |\eta| \, (\operatorname{sech} r)^{\frac{5}{2}} \right| \,, \tag{46}$$

and

$$R - n = \frac{\sinh r \, \left[\cosh r \left| 1 - |\eta| \, (\operatorname{sech} r)^{\frac{5}{2}} \right| - \sinh r \right]}{1 + |\eta|^2 - 2|\eta| \, \sqrt{\operatorname{sech} r}} \,. \tag{47}$$

In the case that $\eta = -1$ the right-hand side of Eq. (47) is plotted as a function of r in Fig. 3. Here we find the maximum negative energy, $R - n \approx 0.3$ at $r \approx 2$. This is slightly less negative energy than can be found in a single squeezed vacuum state. Note that as $r \to 0$, this state becomes $|2\rangle$, a two-particle state with positive energy density everywhere. This is the reason that the behavior in Fig. 3 differs from the $\alpha = 0$ curve in Fig. 2. In the latter case, $\eta = 1$, and there is negative energy for all values of r.

IV. TWO-MODE ENTANGLED STATES

In this section, we will consider several examples of entangled states involving two modes.

A. An Entangled Squeezed State - the Barnett-Radmore State

Our first example of a two-mode entangled squeezed state was described by Barnett and Radmore [18] and is defined by

$$|\psi\rangle = S_{AB}\,|0\rangle\,, \tag{48}$$

where $|0\rangle$ is the vacuum state for both modes, and

$$S_{AB} = \mathrm{e}^{(\xi^* ab - \xi a^\dagger b^\dagger)} \tag{49}$$

is a two-mode squeeze operator. If one were to expand the state $|\psi\rangle$ in terms of number eigenstates, the expansion would contain states with an even total number of particles, with half of these particles in each mode. One has the following identities [18]

$$\begin{aligned}
S_{AB}(-\xi)\,aS_{AB}(\xi) &= a\cosh r - b^\dagger\,\mathrm{e}^{i\delta}\,\sinh r \\
S_{AB}(-\xi)\,a^\dagger S_{AB}(\xi) &= a^\dagger\cosh r - b\,\mathrm{e}^{-i\delta}\,\sinh r \\
S_{AB}(-\xi)\,bS_{AB}(\xi) &= b\cosh r - a^\dagger\,\mathrm{e}^{i\delta}\,\sinh r \\
S_{AB}(-\xi)\,b^\dagger S_{AB}(\xi) &= b^\dagger\cosh r - a\,\mathrm{e}^{-i\delta}\,\sinh r\,.
\end{aligned} \tag{50}$$

Note that

$$S^\dagger_{AB}(\xi) = S^{-1}_{AB}(\xi) = S_{AB}(-\xi)\,. \tag{51}$$

One may use these relations to show that

$$n_1 = n_2 = \sinh^2 r\,, \quad R_1 = R_2 = R_3 = 0\,, \quad R_4 = \sinh r\,\cosh r\,,\ \text{and}\ \gamma_4 = \delta + \pi\,. \tag{52}$$

The minimum energy density in this state is

$$\rho_{\min}(BR) = -\frac{\sinh r}{V}\,\left[2\sqrt{\omega_1\omega_2}\,\cosh r - (\omega_1+\omega_2)\sinh r\right]. \tag{53}$$

This is never more negative than the minimum energy density that would be obtained if the two modes were individually in squeezed vacuum states. The latter energy density is $\rho_{\min}(2SQ) = -(\omega_1+\omega_2)(R-n)/V$, where $R-n$ is given by Eq. (30). Thus we can write

$$\rho_{\min}(BR) - \rho_{\min}(2SQ) = \frac{\sinh r\,\cosh r}{V}\,(\sqrt{\omega_1}-\sqrt{\omega_2})^2\,, \tag{54}$$

which is always non-negative and approached zero only when the two modes have nearly the same frequency.

B. An Second Entangled Squeezed State - the Zhang State

In this subsection, we will consider a second possibility for an entangled two-mode squeezed state, which was discussed by Zhang [15]. This state is defined by

$$|\psi\rangle = N\left(|\bar{\xi}\rangle_a\,|\bar{\eta}\rangle_b + \mathrm{e}^{i\theta}\,|\xi\rangle_a\,|\eta\rangle_b\right)\,, \tag{55}$$

where $|\bar{\xi}\rangle_a$ and $|\xi\rangle_a$ are single-mode squeezed vacuum states for mode a, and $|\bar{\eta}\rangle_b$ and $|\eta\rangle_b$ are such states for mode b. In general, ξ, $\bar{\xi}$, η, and $\bar{\eta}$ can be four arbitrary complex parameters. However, we will restrict our attention to the case where they are real and satisfy

$$\xi = \eta = -\bar{\xi} = -\bar{\eta}\,. \tag{56}$$

In this case,

$$N = \left\{2[1 + \mathrm{Re}(\mathrm{e}^{i\theta}\,\langle -\xi|\xi\rangle_a\,\langle -\eta|\eta\rangle_b)]\right\}^{-\frac{1}{2}} = [2\,(1+\cos\theta\,\mathrm{sech}\,2r)]^{-\frac{1}{2}}\,, \tag{57}$$

where we have used Eq. (A8) for each of the two modes. Similarly, we find

$$n_1 = n_2 = 2\,N^2\,\sinh^2 r\,\left[1 - \frac{\cos\theta}{(\cosh 2r)^{\frac{3}{2}}}\right]\,, \tag{58}$$

and

$$R_1 = R_2 = N^2\,|\sin\theta|\,\frac{\tanh 2r}{\sqrt{\cosh 2r}}\,. \tag{59}$$

Here we have use Eqs. (A4) and (A5), as well as the identity $\sinh^2 r + \cosh^2 r = \cosh(2r)$. In addition, we find $R_3 = R_4 = 0$ and $\gamma_1 = \gamma_2 = -\pi/2$.

In this case, the energy density, Eq. (7), becomes

$$\rho = \frac{1}{V}\left(n_1\,(\omega_1{+}\omega_2){+}R_1\;\{\omega_1\,\cos[2(\mathbf{k}_1\cdot\mathbf{x}-\omega_1 t)+\gamma_1] + \omega_2\,\cos[2(\mathbf{k}_2\cdot\mathbf{x}-\omega_2 t)+\gamma_1]\}\right) \tag{60}$$

We can always choose the spatial position $\mathbf{x}$ and time t so as to make both cosine functions equal to -1, in which case we achieve the minimum allowed energy density in this state of

$$\rho_{min} = -\frac{\omega_1+\omega_2}{V}\,(R_1 - n_1)\,. \tag{61}$$

From Eqs. (57), (58) and (59), we find $R_1 - n_1$ as a function of θ and r. In general, the behavior of this entangled state is similar to that of the superposed squeezed vacua illustrated in Fig. 1. However, there is one limit of particular interest, which is when $r \ll 1$

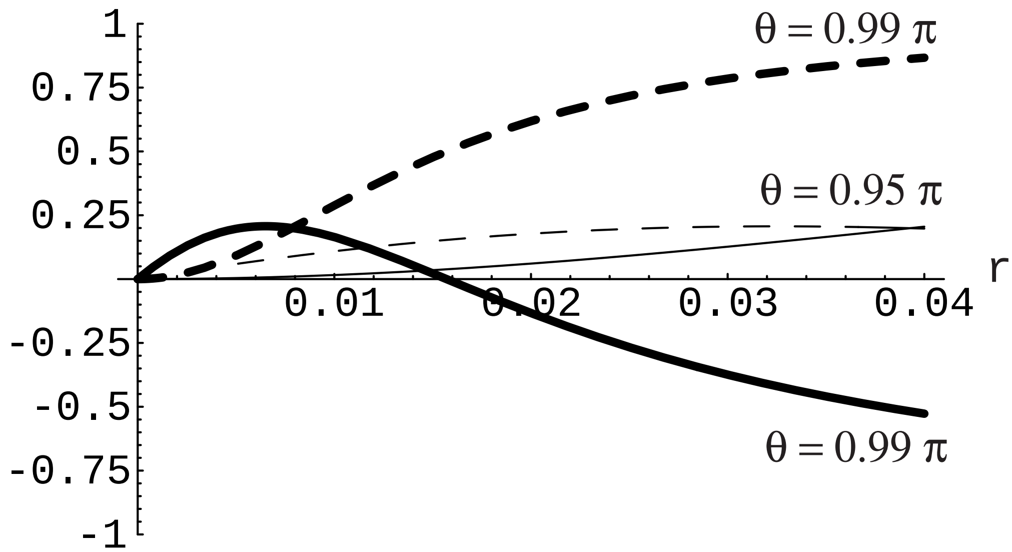

FIG. 4: The quantities $R_1 - n_1$ (solid lines) and n_1 (dashed lines) are plotted for two values of θ as functions of r for the entangled squeezed state defined in Eqs. (55) and (56). In the limit that θ is close to π , one can have appreciable negative energy at small values of r. We see that the peak negative energy, $R_1 - n_1 \approx 0.25$ occurs at $r \approx 0.007$ for $\theta = 0.99\pi$ and at $r \approx 0.035$ for $\theta = 0.95\pi$, whereas the mean particle number is about the same for both cases, $n_1 \approx 0.2$.

and $0 < |\pi - \theta| \ll 1$. (Note that if $\theta = \pi$, then $R_1 = 0$, and there is no negative energy.) If we take the limit $r \ll 1$, for fixed $\theta \neq \pi$, then we find the asymptotic forms

$$n_1 \sim \frac{1 - \cos\theta}{1 + \cos\theta}\, r^2 \,, \tag{62}$$

and

$$R_1 - n_1 \sim \frac{|\sin\theta|}{1 + \cos\theta}\, r \,. \tag{63}$$

In the case that $0 < |\pi - \theta| \ll 1$, the coefficient in the expression for n_1 can be large, so we can have an unusually large particle number in relation to the value of r. The quantities $R_1 - n_1$ and n_1 are plotted in Fig. 4 for two values of θ close to π. In this case, we can have a reasonable amount of negative energy at very small values of the squeeze parameter, r.

C. Entangled Coherent States

In this subsection, we consider a state of the same form as that in Eq. (55), but involving entangled coherent states for two modes, which was also discussed by Zhang [15]. Let

$$|\psi\rangle = N\left(|\alpha\rangle_a\,|\beta\rangle_b + \mathrm{e}^{i\theta}\,|\alpha'\rangle_a\,|\beta'\rangle_b\right)\,, \tag{64}$$

where $|\alpha\rangle_a$, *etc* are single-mode coherent states. We will restrict our attention to the case where the magnitudes of the four complex coherent state parameters are all equal, and $\alpha' = -\alpha$ and $\beta' = -\beta$. Thus

$$|\alpha| = |\beta| = |\alpha'| = |\beta'| = \sigma\,, \tag{65}$$

and

$$\delta_1 - \delta_1' = \pm\pi\,, \quad \delta_2 - \delta_2' = \pm\pi\,. \tag{66}$$

Here $\delta_1, \delta_1', \delta_2, \delta_2'$ are the phases of $\alpha, \alpha', \beta, \beta'$, respectively. In this case, we find

$$N = \left[2(1+\cos\theta\,\mathrm{e}^{-4\sigma^2})\right]^{-\frac{1}{2}}\,, \tag{67}$$

and

$$\begin{aligned} n_1 &= n_2 = 2\sigma^2\,N^2\,(1-\cos\theta\,\mathrm{e}^{-2\sigma^2})\,,\\ R_1 &= R_2 = 2\sigma^2\,N^2\,(1+\cos\theta\,\mathrm{e}^{-2\sigma^2})\,,\\ R_3 &= \sigma^2\,N^2\,(1-\cos\theta\,\mathrm{e}^{-4\sigma^2})\,,\\ R_4 &= \sigma^2\,, \end{aligned} \tag{68}$$

as well as $\gamma_1 = 2\delta_1'$, $\gamma_2 = 2\delta_2'$, $\gamma_3 = \delta_2 - \delta_1$, and $\gamma_4 = \delta_1 + \delta_2$. Let $\phi_1 = \mathbf{k_1}\cdot\mathbf{x} - \omega_1\mathbf{t}$ and $\phi_2 = \mathbf{k_2}\cdot\mathbf{x} - \omega_2\mathbf{t}$. We then set

$$\phi_1 + \delta_1 = \phi_2 + \delta_2 = \frac{\pi}{2}\,, \tag{69}$$

which can always be done by a suitable choice of $\mathbf{x}$ and t. The energy density for a two-mode traveling wave state, Eq. (7) now becomes

$$\rho = \frac{1}{V}\left[n_1\omega_1 + n_2\omega_2 - R_1\omega_1 - R_2\omega_2 + (R_3 - R_4)\sqrt{\omega_1\omega_2}\,(1+\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)\right]. \tag{70}$$

If the two modes are close in wavenumber, so that $\omega_1 \approx \omega_2 = \omega$ and $\hat{\mathbf{k}}_\mathbf{1} \approx \hat{\mathbf{k}}_\mathbf{2}$, and we set $\theta = 0$ then

$$\rho = -\frac{4\,\omega}{V}\,f(\sigma)\,, \tag{71}$$

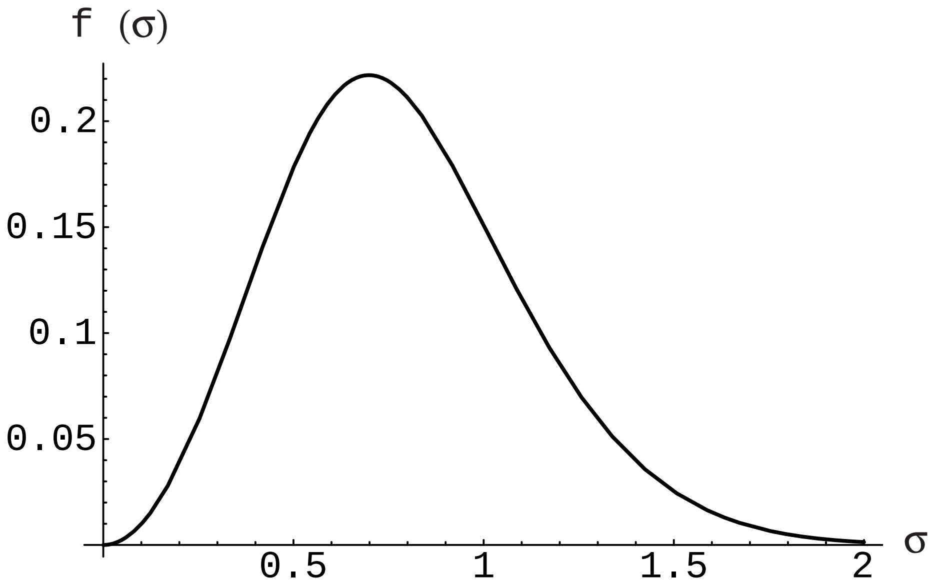

FIG. 5: The function $f(\sigma)$, given by Eq. (72), is plotted. Its maximum, at $\sigma \approx 0.7$, describes the case of maximal negative energy density for the entangled coherent state defined in Eqs. (64), (65) and (66).

where

$$f(\sigma) = \frac{\sigma^2\, \mathrm{e}^{-2\sigma^2}\, (1 + \mathrm{e}^{-2\sigma^2})}{1 + \mathrm{e}^{-4\sigma^2}}\,. \tag{72}$$

The function $f(\sigma)$ is plotted in Fig. 5, where we see that it attains a maximum value of about 0.22 at $\sigma \approx 0.7$. This corresponds to a negative energy density of $\rho \approx -0.88\omega/V$, which is about three times as negative as the maximally negative energy density found in the superposed coherent states discussed in Sect. III A.

V. SUMMARY

In this paper we have developed a formalism for parameterizing the energy density in states of a massless scalar field in which either one or two modes are excited. We found explicit expressions for the energy density for the cases of traveling waves and of standing waves in one spatial direction. In all cases, the maximum negative energy density which can be achieved in a given state can be expressed in terms of our parameters.

We next applied this approach to find the maximum negative energy density in several states, including some states which are of current interest in quantum optics. For the case

of a single mode, we considered three possible superposition states: (1) two coherent states, (2) two squeezed vacuum states, and (3) a coherent state and a squeezed vacuum state. The superposition of two coherent states can be described as a Schrödinger "cat state" in the sense that it would be a superposition of two classical configurations in the limit of large coherent state parameter. Here we find that the maximal negative energy density is achieved with mean photon numbers slightly less than one. This is an example where a quantum superposition state has negative energy density, even though each component of the superposition would have positive energy density by itself. In the case of the superposition of two squeezed vacuum states, one finds the opposite effect. Although here the superposition state does has negative energy density, it is somewhat less negative than in the case of a single squeezed vacuum state. Furthermore, the most negative energy density now occurs for small mean photon number, as opposed to large number in the case of a single squeezed vacuum state. In the case of a superposition of a coherent state and a squeezed vacuum state, we find that for fixed coherent state parameter, there is negative energy density for small squeeze parameter, and again for larger values, but there is an intermediate range where the energy density is always positive.

We next examined some two-mode states involving entanglement between the two modes, including two examples of entangled squeezed vacuum states. The first example, the Barnett-Radmore state [18], exhibits somewhat less negative energy density than would be found if each mode were separately in a squeezed vacuum state. In the second example, the Zhang state [15], we find results similar to those in the superposition of squeezed vacuum states. There is negative energy in the Zhang state, but only for small mean particle numbers. Finally, we examined a two-mode entangled coherent state, which also exhibits negative energy for small mean particle number. It is also similar to the case of a superposition of coherent states of a single mode, but the entangled state has somewhat more negative energy density.

One of the motivations for this investigation is to draw links between theoretical studies of violations of the weak energy condition, and experimental work in quantum optics. We hope that this line of work will lead to further experimental studies of subvacuum phenomena.

Acknowledgments

We would like to thank Piotr Marecki for useful discussions. This work was supported in part by the National Science Foundation under Grant PHY-0555754 to LHF.

APPENDIX A

In this appendix, we will calculate some of the matrix elements of operators such as $a^\dagger\, a$ and a^2 which are needed to find the energy density in the states treated in this paper. We begin with matrix elements between squeezed vacuum states. The diagonal matrix elements $\langle\xi|a^\dagger\, a|\xi\rangle$ and $\langle\xi|a^2|\xi\rangle$ are given by Eq. (29). We need off-diagonal matrix elements of the form $\langle-\xi|a^\dagger\, a|\xi\rangle$ and $\langle-\xi|a^2|\xi\rangle$, where ξ is real. If we set $\delta = 0$, so that $\xi = r$, then Eq. (27) becomes

$$|\xi\rangle = \sqrt{\mathrm{sech}r}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{n!}\Big(-\frac{1}{2}\mathrm{tanh}r\Big)^n\,|2n\rangle\,. \tag{A1}$$

This leads to the result

$$\langle-\xi|a^\dagger a|\xi\rangle = \langle\xi|a^\dagger a|-\xi\rangle = 2\,\mathrm{sech}r\sum_{n=1}^{\infty}\frac{(2n)!}{n!(n-1)!}\,(-1)^n\Big(\frac{1}{2}\mathrm{tanh}r\Big)^{2n}\,. \tag{A2}$$

Use the fact that

$$\sum_{n=1}^{\infty}\frac{(2n)!}{n!(n-1)!}\,(-1)^n\left(\frac{1}{2}x\right)^{(2n-2)} = -\frac{2}{(1+x^2)^{3/2}}\,, \tag{A3}$$

to find

$$\langle-\xi|a^\dagger a|\xi\rangle = \langle\xi|a^\dagger a|-\xi\rangle = -\frac{\mathrm{sech}r\,\mathrm{tanh}^2 r}{(1+\mathrm{tanh}^2 r)^{3/2}}\,. \tag{A4}$$

Similarly, we may use Eq. (A1) to show that

$$\langle-\xi|a^2|\xi\rangle = -\langle\xi|a^2|-\xi\rangle = \mathrm{sech}r\sum_{n=1}^{\infty}\frac{(2n)!}{n!(n-1)!}\,(-1)^n\left(\frac{1}{2}\mathrm{tanh}r\right)^{(2n-1)} = -\frac{\mathrm{sech}r\,\mathrm{tanh}r}{(1+\mathrm{tanh}r)^{3/2}}\,. \tag{A5}$$

Because these matrix elements are real, we have that

$$\langle-\xi|(a^\dagger)^2|\xi\rangle = \langle\xi|(a^\dagger)^2|-\xi\rangle = \langle-\xi|a^2|\xi\rangle\,. \tag{A6}$$

In the present case, the squeeze operator is

$$S(\xi) = S(r) = \mathrm{e}^{\frac{1}{2}r[a^2-(a^\dagger)^2]}\,. \tag{A7}$$

From this relation, we see that

$$\langle -\xi|\xi\rangle = \langle\xi|-\xi\rangle = \langle 0|S^2(r)|0\rangle = \langle 0|S(2r)|0\rangle = \sqrt{\mathrm{sech}(2r)}\,. \tag{A8}$$

Next we derive the matrix elements involving both a coherent state and a squeezed vacuum state that are needed in Sect. III C. A coherent state may be represented in terms of number eigenstates as [18]

$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sum_{\ell=0}^{\infty}\frac{\alpha^\ell}{\sqrt{\ell!}}\,|\ell\rangle\,. \tag{A9}$$

This may be combined with Eq. (27) to show that

$$\begin{aligned}\langle\xi|\alpha\rangle = \langle\alpha|\xi\rangle^* \;&=\; \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{n!}\,[-\frac{1}{2}\mathrm{e}^{-i\delta}\tanh r]^n \\ &=\; \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\exp\left(-\frac{1}{2}\mathrm{e}^{-i\delta}\alpha^2\tanh r\right)\,,\end{aligned} \tag{A10}$$

and

$$\langle\xi|a^\dagger a|\alpha\rangle = \langle\alpha|a^\dagger a|\xi\rangle^* = -\mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\mathrm{e}^{-i\delta}\alpha^2\tanh r\,\exp\left(-\frac{1}{2}\mathrm{e}^{-i\delta}\alpha^2\tanh r\right)\,. \tag{A11}$$

Note that

$$\langle\xi|a^2|\alpha\rangle = \langle\alpha|(a^\dagger)^2|\xi\rangle^* = \alpha^2\,\langle\xi|\alpha\rangle\,. \tag{A12}$$

Finally, we show that

$$\begin{aligned}\langle\xi|(a^\dagger)^2|\alpha\rangle \;&=\; \langle\alpha|a^2|\xi\rangle^* = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\sum_{n=1}^{\infty}\frac{2n(2n-1)}{n!}\,\alpha^{2n-2}\,\left(-\frac{1}{2}\mathrm{e}^{-i\delta}\tanh r\right)^n \\ &=\; \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\,\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{n!}\,\left(-\frac{1}{2}\mathrm{e}^{-i\delta}\tanh r\right)^n \\ &=\; \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\,\exp\left(-\frac{1}{2}\mathrm{e}^{-i\delta}\alpha^2\tanh r\right) \\ &=\; \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\,\sqrt{\mathrm{sech}r}\,\left(\mathrm{e}^{-i\delta}\alpha^2\tanh r-1\right)\,\mathrm{e}^{-i\delta}\,\tanh r \\ &\times\,\exp\left(-\frac{1}{2}\mathrm{e}^{-i\delta}\alpha^2\tanh r\right)\,.\end{aligned} \tag{A13}$$

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